Spot it!

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Quick! Find the matching symbol between the two cards!

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Each card in a Spot It! Deck has 8 different symbols. The game is played on the premise that between any two cards there is one—and only one—matching symbol.

How is this possible? Is there a way to generalize the Spot it! deck to create decks of different sizes? In what sizes does this game exist?

I. Basic games

First, let's build a deck with two symbols per card:

Notice that between any two cards there is exactly one match. By representing symbols as letters, we'll show the two symbol deck in this way:

AB AC BC

Here's a deck with three symbols per card:

ABC BDF CDG
ADE BEG CEF
AFG

For a small number of symbols per card it seems simple to construct a deck that can be used to play the game. Is there an easy way to construct or show the existence of larger decks ?

We discovered a deck can be constructed of any size as long as the number of symbols per card is a prime power, p^k, plus one. We will use projective geometry to show how.

II. Finite Projective Planes

In a projective plane, every line has n points and every pair of lines meets in exactly one point. If we say points represent symbols and lines represent cards, then a projective plane has the exact properties we want a deck to have. For example, Figure 1 corresponds the deck with three symbols per card. Projective planes ensure that any two cards have exactly one symbol in common.

It turns out a finite projective plane is an exact representation of a deck of Spot it! cards. So the existence of a game depends on the existence of a corresponding projective plane.

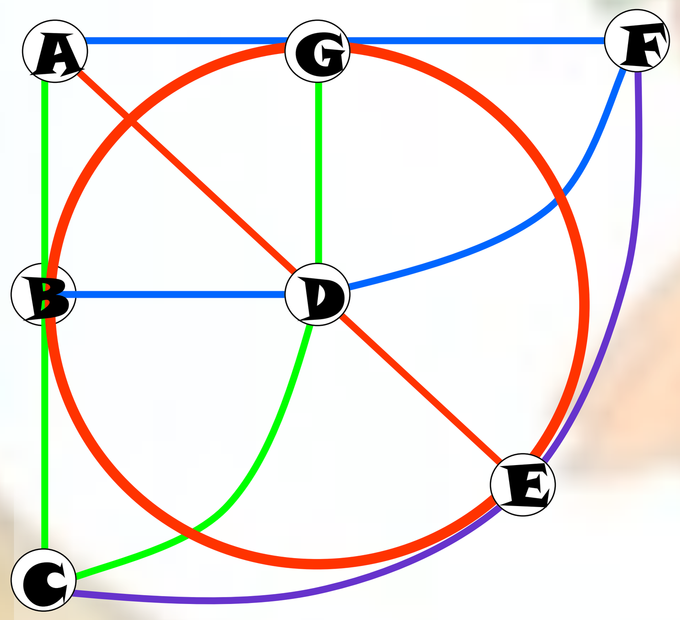

Figure 1. A complete finite projective plane of order 2, representing a Spot it! game with three symbols per card. The blue lines are the cards AGF and BDF, which have the match F.

III. Completing an Affine Plane to construct a finite projecting plane

A projective plane of order n exists if and only if an affine plane of order n exists. Only affine planes of order p^k exist. Order p^k means the plane has p^{2k} points, $p^{2k} + p^k$ lines, and each line contains exactly p^k points. The smallest affine plane is shown below:

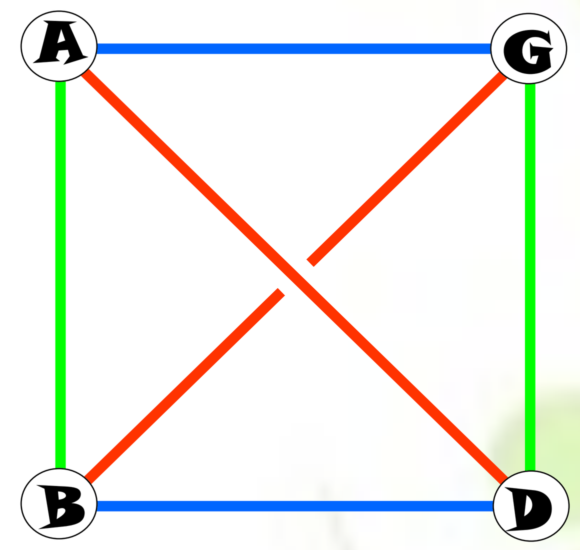

Figure 2. Order 2 affine plane. Even though the red lines appear to cross, they are parallel because they have no points in common.

Any affine plane can be completed to a projective plane by follow these steps:

- Add one point for each set of parallel lines. There will always be p^k points added.
- Extend parallel lines to the new point. (In affine geometry, parallel is defined as line with no points in common.)
- Form a new line with the new points. This makes $p^{2k} + p^k + 1$ lines, each with $p^k + 1$ points.

Figure 1 is the completed order 2 affine plane from Figure 2. Notice the blue lines are parallel in the affine plane, and are both extended to point F in the projective plane. The same is done for the parallel red and green lines.

IV. Constructing a deck using Latin Squares

A Latin square of order n is an arrangement of n x n cells containing symbols, each of which occurs exactly once in every row and every column.

A Latin square is mutually orthogonal to another if no corresponding cells between the two contain the same ordered symbols as other corresponding cells.

The existence of mutually orthogonal Latin squares (MOLS) is connected to the existence of affine and projective planes, and thus Spot it! decks. The following is a step by step process showing the construction of a deck from the equivalent MOLS as well as the association of MOLS with affine and projective planes.

For any number p^k, there are exactly $p^k - 1$ MOLS. For any $p^k - 1$ MOLS, there exists a deck with $p^k + 1$ symbols per card.

1. Given a set of $p^k - 1$ MOLS, match the symbols in each corresponding cell on put them on one card together. This will produce p^k cards with $p^k - 1$ symbols per card.

							→			
A	B	C		D	E	F		AD	BE	CF
B	C	A		F	D	E	→	BF	CD	AE
C	A	B		E	F	D		CE	AF	BD

For p^k =3, two MOLS can be made and combined to form the beginning of nine cards.

2. Create p^k new cards--one using every symbol from the first MOLS, one using every symbol from the second MOLS, and so on for all the MOLS. Finally, create one with p^k new symbols.

ABC DEF GHI

Beginnings of three new cards. Now our deck has 12 cards.

3. We define parallel cards as cards that have no symbols in common. Find the parallel sets from the cards in the step 1. To each card in a set add one symbol from the card with all new symbols.

ADG	BEG	CFG	ABC
BFH	CDH	AEH	DEF
CEI	AFI	BDI	GHI

Parallel cards from step one with one new symbol added and three new cards from step 2. These cards correspond to an affine plane.

4. To complete our deck, add one more new card with $p^k + 1$ new symbols. Again find the sets of parallel cards from the affine deck and add one of these new symbols to each card in a set.

ADGJ	BEGK	CFGL	ABCM	JKLM
BFHJ	CDHK	AEHL	DEFM	
CEIJ	AFIK	BDIL	GHIM	

Completed deck with 4 symbols per card. One new card and parallel cards from step three with one new symbol added. These cards correspond to an projective plane.

V. Conclusion

We found that it is possible to make a Spot it! deck with exactly $p^k + 1$ symbols per card, and it can be done using the MOLS method.

Reference:

Lars Kadison and Matthias T. Kromann. Projective Geometry and Modern Algebra. Kirkhäuser Boston Inc., Boston, MA, 1996.